A Multicarrier System Based on the Fractional Fourier Transform for Time–Frequency-Selective Channels

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***Abstract*—Traditional multicarrier techniques perform a frequency-domain decomposition of a channel characterized by frequency-selective distortion in a plurality of subchannels that are affected by frequency flat distortion. The distortion in each independent subchannel can then be easily compensated by simple gain and phase adjustments. Typically, digital Fourier transform schemes make the implementation of the multicarrier system feasible and attractive with respect to single-carrier systems. However, when the channel is doubly selective (that is, time–frequency-selective), as it usually happens in the rapidly fading wireless channel, this traditional methodology fails. Since the channel frequency response is rapidly time-varying, the optimal transmission/reception methodology should be able to process nonstationary signals. In other words, the subchannel carrier frequencies should be time-varying and ideally decompose the frequency distortion of the channel perfectly at any instant in time. However, this ideally optimal approach presents significant challenges both in terms of conceptual and computational complexity. The idea disclosed in this work is that a nonstationary approach can be approximated using signal bases that are especially suited for the analysis/synthesis of nonstationary signals. We propose in fact the use of a multicarrier system that employs orthogonal signal bases of the chirp type that in practice correspond to the fractional Fourier transform signal basis. The significance of the methodology relies on the important practical consideration that analysis/synthesis methods of the fractional Fourier type can be implemented with a complexity that is equivalent to the traditional fast Fourier transform.**

***Index Terms*—Discrete Fourier transform, discrete fractional Fourier transform, land mobile radio cellular systems, orthogonal frequency-division multiplexing, time-varying frequency-selective channels.**

I. Introduction

MULTICARRIER techniques transmit data by dividing the stream into several parallel bit streams. Each of the subchannels has a much lower bit rate and is modulated onto a different carrier. Orthogonal frequency-division multiplexing (OFDM) is a special case of multicarrier modulation with equally spaced subcarriers and overlapping spectra [1]. The OFDM time-domain waveforms are chosen such that mutual orthogonality is ensured in the frequency domain. Time dispersion is easily handled by such systems because the substreams are essentially free of intersymbol interference (ISI). This last aspect of multicarrier schemes has contributed significantly to increase the popularity of OFDM in many wireless and wire-line applications [2], [3], [17] and has opened a "competition" with more traditional single-carrier time-domain schemes [5], [4], [16]. When the channel is, however, frequency-dispersive (that is, it is afflicted by nonnegligible Doppler spread as in the wireless channel), interchannel interference may degrade an OFDM system performance to untolerable levels. Consider the baseband equivalent signal generated by a generic N-channel multicarrier system expressed as

$$\begin{aligned} s(t) &= \sum_{k=-\infty}^{+\infty} \sum_{l=0}^{N-1} a_{k,l} g_{k,l}(t) \\ &= \sum_{k=-\infty}^{+\infty} \sum_{l=0}^{N-1} a_{k,l} \phi_l(t-kT) \end{aligned} \tag{1}$$

where T is the symbol period, $a_{k,l}$ is the (generally complex valued) information bearing symbol, and $g_{k,l}(t)=\phi_l(t-kT)$, $l=0, 1, \ldots, N-1$ are the fundamental basis waveforms. In wireless links, the transmitted signal $s(t)$ is linearly distorted by the multipath fading channel operator[1] $\mathbf{H}$ as

$$\begin{aligned} y(t) = (\mathbf{H}s)(t) &= \sum_{k=-\infty}^{+\infty} \sum_{l=0}^{N-1} a_{k,l} f_{k,l}(t) \\ &= \sum_{k=-\infty}^{+\infty} \sum_{l=0}^{N-1} a_{k,l} f_l(t-kT) \end{aligned} \tag{2}$$

where $f_l(t)=(\mathbf{H}\phi_l)(t)$. The fundamental problem is to select the transmission basis $\phi_l(t)$ in such a way that the projection of the signal onto an identically structured signal set $f_{k,l}(t)$ gives the transmitted symbols as

$$\int_{-\infty}^{+\infty} y(t) f_{k,l}^*(t)\,dt = \int_{-\infty}^{+\infty} y(t) f_l^*(t-kT)\,dt = a_{k,l}.$$

This condition implies relative simplicity of the receiver and robustness to additive white Gaussian noise.

The selection of a channel-dependent signal basis in the case of a static channel $(\mathbf{H}s)(t) = \int h(\tau)s(t-\tau)\,d\tau$ is well

[1]The operator $\mathbf{H}$ represents the effects of multipath RF waves propagation. One standard way of representing the channel is

$$(\mathbf{H}s)(t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} S(\tau, \nu) s(t-\tau) e^{j2\pi\nu t}\,d\tau\,d\nu$$

where $S(\tau, \nu)$ is the spreading function of the wireless channel [8].

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understood and corresponds to functions $\phi_l(t)$ equal to the eigenmodes of the channel autocorrelation. The eigenmodes of a frequency-selective static channel can be well approximated by Fourier bases and transmitter and receiver easily implemented by means of Fourier transform methods. The popularity of OFDM schemes stems exactly from this fundamental property. It is important to realize that, if the channel is doubly dispersive, as in the rapidly fading wireless channel, the entire conceptual framework of the basic Fourier-domain channel partitioning scheme loses its optimality. A similar point of view was expressed in [9], where orthogonality was somehow sacrificed for time–frequency localization of the transmitted signal set: an increased robustness to doubly dispersive channel distortions was the main goal. However, the problem was not attacked at the root, because exponential (Fourier-like) signal sets were still used, both at the transmitter and at the receiver. In this work, we introduce a new methodology that employs a signal set specifically designed for the synthesis/analysis of time-varying (nonstationary) signals.

We can think of the doubly dispersive channel as a channel with "time-varying" frequency response. The optimal transmission/reception methodology should be able to "diagonalize" nonstationary signals (in a sense employing Karhunen–Loeve eigenfunctions as bases). Roughly speaking, the subchannel carrier frequencies should be time-varying and ideally decompose the frequency distortion of the channel perfectly at any instant in time. This optimal approach presents significant challenges both in terms of conceptual and computational complexity.

The idea disclosed in this work is that the optimality of a nonstationary approach can be somehow *approximated*, without any increase of computational complexity with respect to the classical OFDM scheme. We propose the use of a multicarrier system that employs orthogonal signal bases of the chirp type. Such bases are related to the fractional Fourier transform whose time–frequency properties [10] are well known in the signal processing community. Exactly as the Fourier harmonic analysis employs sinusoidal function to decompose periodic signals, fractional Fourier techniques employ chirp "harmonics" for the decomposition of signals with time-varying periodicity. The significance of the methodology we introduce relies on the important practical consideration that analysis/synthesis methods of the fractional Fourier type are implemented with a complexity that is equivalent to traditional fast Fourier transform (FFT) computational procedures. The paper is organized as follows. In Section II, we introduce the fractional Fourier transform. In Section III, we describe the system model and describe the key realization of the transceiver. In Section IV, we show a discrete-time (digital) implementation of the idea, and in Section V we perform some experiments that compare the proposed technology with the traditional OFDM system.

II. The Fractional Fourier Transform

The fractional Fourier transform (FRFT) was introduced in [6] and [7] as a generalization of the Fourier transform. The transform immediately appeared useful in many signal processing applications, but remained essentially unknown until recently, when several new results, interpretations and applications have been found [11], [10]. The interpretation of the FRFT as a "rotation" operator in the time–frequency plane [10] has generated significant interest in the signal processing community. The transformation kernel of the continuous FRFT is defined as follows:

$$K_\alpha(t, u) = A_\alpha e^{j\pi(t^2+u^2)\cot\alpha - j2\pi tu\csc\alpha} \tag{3}$$

where α is the rotation angle of the transformed signal and

$$A_\alpha = \frac{e^{\{-j\pi \mathrm{sign}[\sin\alpha]/4 + j\alpha/2\}}}{\sqrt{|\sin\alpha|}}.$$

The forward and inverse FRFT are defined as

$$\mathcal{F}_\alpha\{x(t)\}(u) = \mathcal{X}_\alpha(u) = \int_{-\infty}^{+\infty} x(t)K_\alpha(t, u)\,dt \tag{4}$$

$$x(t) = \int_{-\infty}^{+\infty} \mathcal{X}_\alpha(u)K_{-\alpha}(t, u)\,du. \tag{5}$$

The domains of the signal for $0 < |\alpha| < \pi$ are defined *fractional Fourier domains* and $\alpha = \pi/2$ in (4) and (5) gives the traditional Fourier transform (FT). The FT can be considered a projection of a given signal in the time–frequency plane on the frequency axis, or, in other words, a rotation of $\pi/2$ with respect to the time axis. The FRFT can be considered a projection of the signal on an axis which forms an angle α with the time axis: a rotation in the time–frequency plane. The reader interested in more interpretations and explanations regarding the FRFT transformation is directed to [10].

The problem of interest in most recent literature dealing with the FRFT is the research of efficient digital methods for the computation of the transforms in (4) and (5). In [11], a method with $N \log N$ (N is the number of samples) complexity was recently discovered that also provides significant accuracy of representation. This interesting result opens the possibility of using the FRFT at no extra cost anywhere traditional FFT schemes reveal their inefficiency in dealing with nonstationary signals. Shortly thereafter, the same authors [12] have presented a method of filtering in the fractional Fourier domain equivalent to the traditional Fourier domain filter. While Fourier domain filters have the dual time-domain interpretation as convolutions, fractional Fourier domain filters correspond to complicated operations of time and frequency. Nevertheless, it was shown in [12] that in many cases this type of filtering achieves a mean square error which is lower than traditional Fourier domain filters. This is particularly true when recovery from nonstationary distortions is attempted.

Traditional techniques for multicarrier modulation attempt a frequency decomposition of the transmission available bandwidth. The effect of the time-invariant channel distortions can be compensated on a subchannel-by-subchannel basis by single-tap frequency-domain equalizers. As a consequence, the overall traditional multicarrier system can be seen as an optimal Fourier-domain filter. However, if the channel is time-varying, the traditional multicarrier system loses optimality because the optimal recovery operator is in general time-variant. This means that it cannot be implemented in the conventional

Fourier domain and is exactly the reason that motivates the use of an FRFT-based technique.

III. System Model

Consider the baseband representation of a multicarrier system as in (1) with

$$g_{k,\,l}(t) = g(t - kT)e^{j2\pi lFt} \tag{6}$$

where F is the carrier frequency spacing. The use of pulses as in (6) results in a rectangular *tiling* of the time–frequency plane. The product $TF \geq 1$ defines the time–frequency product of each independent function in the signal set. In the OFDM case, the pulse $g(t)$ in (6) is a rectangular window of duration T and $F = 1/T$. A *coarsification* of the time–frequency grid is typically employed using a guard-time between temporal adjacent symbols for mitigation of the time-dispersive characteristic of a frequency-selective channel. On the other hand, properly shaping the basic symbols in each subchannel by using a pulse different from the rectangular one mitigates frequency dispersion effects on the channel caused by Doppler spreads. This approach is, however, fundamentally suboptimum and likely to be ineffective in the presence of large Doppler spreads. Suppose that we decided to use a signal basis that does not operate a rectangular tiling of the time–frequency plane, for example, [13]

$$g_{k,\,l}(t) = \phi_l(t - kT) = g(t - kT)f_{-\alpha,\,l}(t - kT) \tag{7}$$

where $f_{\alpha,\,n}(t)$ is given by

$$\begin{aligned} f_{\alpha,\,n}(t) &= \sqrt{\frac{\sin\alpha + j\cos\alpha}{T}} \\ &\quad \times \exp\left(-j\frac{t^2 + (\sin(\alpha)n2\pi/T)^2}{2}\cot\alpha + jn(2\pi/T)t\right), \\ &\quad -T/2 \leq t \leq T/2. \end{aligned} \tag{8}$$

The selection of the function $f_{\alpha,\,n}(t)$ is motivated by the fact that exactly as the exponential (sinusoidal) basis of the FT corresponds in the Fourier domain to an impulse, the basis $f_{\alpha,\,n}(t)$ corresponds (up to a constant factor) in the Fractional Fourier domain to an impulse.

In other words [13], $f_{\alpha,\,n}(t)$ has been derived in such a way that

$$\begin{aligned} &FRFT_{-\alpha}[\delta(u - n(\sin\alpha/T))] \\ &\quad = \text{const} \times f_{\alpha,\,n}(t), \\ &\qquad n = -\infty, \ldots, -1, 0, 1, \ldots +\infty. \end{aligned}$$

The derivation of this interesting fact is reported in [13]. It is also important to observe that the set of functions $f_{\alpha,\,l}(t)$ is an orthonormal set. In fact, we have [13]

$$\int_{-\infty}^{+\infty} f_{\alpha,\,m}(t)f^*_{\alpha,\,n}(t)\,dt = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n. \end{cases} \tag{9}$$

In Figs. 1 and 2, we show two members of the set $f_{\alpha,\,n}(t)$ for $\alpha = \pi/2\ 100\,000$ and $N = 256$ sampled at 10 kHz. In Fig. 3, we show spectral energy distributions of the two functions.

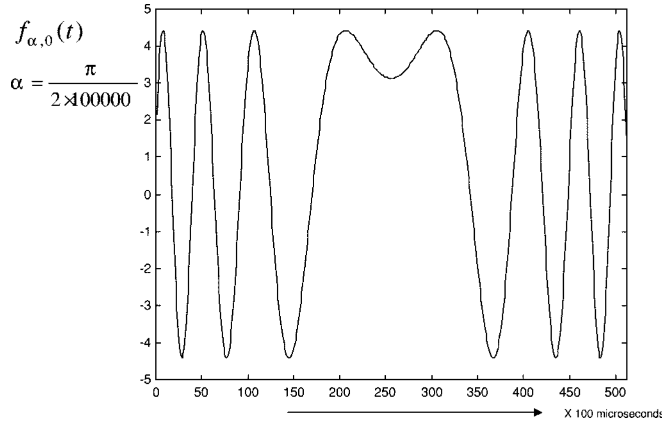

Fig. 1. The basis waveform $f_{\alpha,\,n}(t)$ with $N = 256$ for $n = 0$.

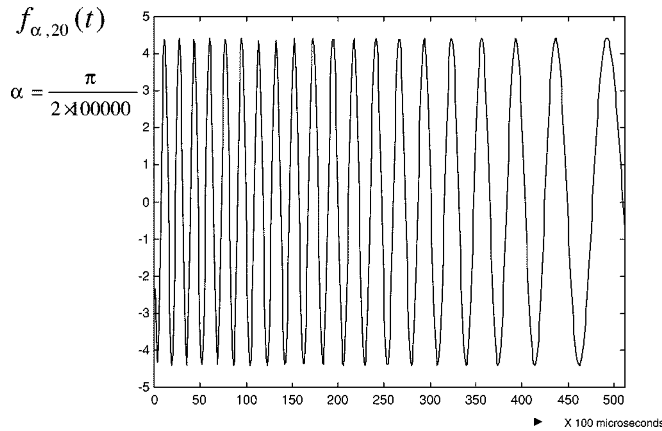

Fig. 2. The basis waveform $f_{\alpha,\,n}(t)$ with $N = 256$ for $n = 20$.

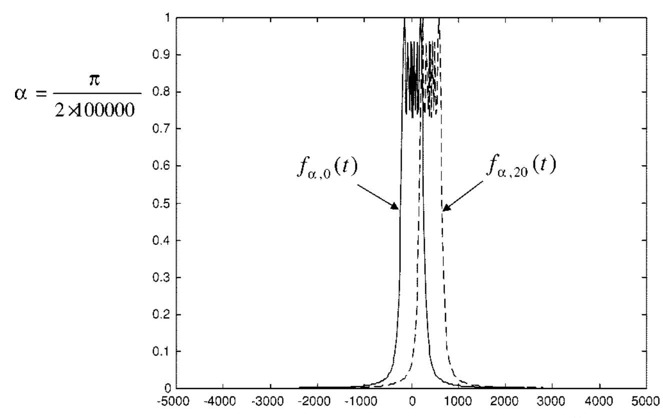

Fig. 3. Spectral energy distribution of the two basis waveforms shown in Figs. 1 and 2.

If the pulse $g(t)$ is a rectangular window of duration T, the transmission system (1) with (7) is equivalent to a new method of modulation where the generic basis signal is a chirp signal with chirp rate $-\cot\alpha$ and the frequency spacing between adjacent basis signals is $1/T$.

The frequency of the nth basis signal is (linearly) dependent on time and is equal to

$$\omega_{\alpha,\,n} = n\frac{2\pi}{T} - t\cot\alpha.$$

Traditional OFDM systems are just special cases of (1) with (7) and $\alpha = \pi/2$.

The condition (9) shows that the transmitted symbols (in the absence of channel distortions) can be recovered by performing the following matched filtering operation:

$$\begin{aligned} a_{k,\,l} &= \int_{-\infty}^{+\infty} s(t) g_{k,\,l}^{*}(t)\, dt \\ &= \int_{-\infty}^{+\infty} s(t) g^{*}(t-kT) f_{\alpha,\,l}^{*}(t-kT)\, dt. \end{aligned} \tag{10}$$

A. The Transceiver as a Fractional Fourier Domain Filter

Assume that a single-shot transmission for the kth block of symbols has to be achieved and that the number of subchannels (carriers) is infinite.[2] Define $\mathcal{F}_{\alpha}\{x(t)\}(u)$ as the continuous FRFT of $x(t)$ with angle equal to α and $s_k(t)$ as the modulated signal $s(t)$ relative to the symbols in the kth block $a_{k,\,-\infty}, \ldots, a_{k,\,+\infty}$

$$s_k(t) = \sum_{n=-\infty}^{+\infty} a_{k,\,n} f_{-\alpha,\,n}(t) = \mathcal{D}_{-\alpha}\{\mathbf{a}_k\}(t) \tag{11}$$

where the symbols in the kth block are organized in a vector $\mathbf{a}_k$. It was shown in [13] that

$$a_{k,\,n} = \mathcal{K} \times \mathcal{F}_{\alpha}\{s_k(t)\} \left(\frac{n}{T} \sin\alpha\right) \tag{12}$$

in (11) with $\mathcal{K}$ as a real constant. We can then interpret the transmission system (1) with (7) as a block inverse FRFT and the receiver as a block FRFT. This means that at the output of the receiver a multiplicative filter of the type described in [12] can be used in the presence of arbitrary time-varying distortion for the channel $\mathbf{H}$. The signal $s_k(t)$ undergoes linear distortion and additive noise $n_k(t)$ as

$$r_k(t) = (\mathbf{H}s_k)(t) + n_k(t).$$

Since, in general, the channel distortion is time-varying (because the kernel of the degadation model represented by the spreading function is nonstatic), the optimal recovery operator for the signal $s_k(t)$ is also a general linear time-varying estimate of the form

$$\hat{s}_k(t) = (\mathbf{W}r_k)(t) = \int_{-\infty}^{+\infty} w(t,\,\tau) r_k(\tau)\, d\tau. \tag{13}$$

The mean square error for nonstationary signals that have finite energy is defined as

$$\mathrm{MSE}(\mathbf{W}) = E\{\|s_k(t) - \hat{s}_k(t)\|^2\}$$

where $\|x(t)\|^2 = \int_{-\infty}^{+\infty} x(t)x^{*}(t)\, dt$. Minimization of $\mathrm{MSE}(\mathbf{W})$ can be obtained by the operator kernel $w(t,\,\tau)$ that satisfies

$$E\{s_k(t) r_k^{*}(\tau)\} = \int_{-\infty}^{+\infty} w(t,\,u) E\{r_k(u) r_k^{*}(\tau)\}\, du. \tag{14}$$

[2]This is an ideal abstraction that helps the derivation of the method and that was also motivated in [9].

The solution of this equation is computationally intensive and not practical. The idea proposed in [12] is the use of a "multiplicative" filter that multiplies the signal $\mathcal{F}_{\alpha}\{r_k(t)\}(u)$ in the αth fractional Fourier domain, so that

$$\hat{s}_k(t) = (\mathbf{W}r_k)(t) = \mathcal{F}_{-\alpha}\{w_{\alpha}(u)\mathcal{F}_{\alpha}\{r_k(t)\}(u)\}(t). \tag{15}$$

It is important to recognize that the class of the αth fractional Fourier filters in (15) is a subclass of the filters in (13). The fractional Fourier domain filters are well suited for the recovery of time-varying degradations and are much more effective [12] than traditional time-invariant techniques.

In practice, due to (12), there is no need to perform the inverse transformation, because the transmitted symbols can be considered samples of the continuous FRFT. We can then write

$$\begin{aligned} &\hat{a}_{n,\,k} \\ &= \mathcal{K}^{-1}[w_{\alpha}(u)\mathcal{F}_{\alpha}\{r_k(t)\}(u)]_{u=(n/T)\sin\alpha} \\ &= \mathcal{K}^{-1}[w_{\alpha}(u)\mathcal{F}_{\alpha}\{(\mathbf{H}\mathcal{D}_{-\alpha}\{\mathbf{a}_k\})(t) + n_k(t)\}(u)]_{u=(n/T)\sin\alpha}. \end{aligned} \tag{16}$$

The block diagram of the system that implements the operations in (16) is shown in Fig. 4.

The solution to the filter estimation problem can be obtained using the calculus of variations method described in [12]. Following this method, we obtain

$$w_{\alpha}(u) = \frac{E\{s_k(u) r_k^{*}(u)\}}{E\{r_k(u) r_k^{*}(u)\}}.$$

It is interesting that the optimal value of α (which defines in what fractional Fourier domain transmission will occur) can be found from the off-line minimization of the following cost function [12]:

$$\begin{aligned} J = \int_{-\infty}^{+\infty} & E\{s_k(u) s_k^{*}(u)\} \\ & - 2\mathrm{Re}\{w_{\alpha}^{*}(u) E\{s_k(u) r_k^{*}(u)\}\} \\ & + w_{\alpha}(u) w_{\alpha}^{*}(u) E\{r_k(u) r_k^{*}(u)\}\, du. \end{aligned} \tag{17}$$

In practice, selecting the angle α that minimizes (17) is equivalent to selecting the fractional Fourier domain that best matches the time-varying characteristics of the channel. Therefore, this formulation also reduces to a classical best-basis selection. It has been suggested by a reviewer that in practice some channels may require α to vary (Doppler spread scenarios may vary depending on relative motion between the transmitter and receiver). While we briefly describe a method for the estimation of α on a frame-by-frame basis in Section V, the study of an efficient online procedure for the computation of α is an interesting ramification of our work that we have not addressed.

IV. Discrete-Time Implementation

There are many possible ways to discretize the operations in (16). We select the methodology proposed in [11], because it is attractive from the computational point of view. We will start our research of a discrete-time operator for the operations in (16) from the derivation of the discrete-time model for an FFT-based system.

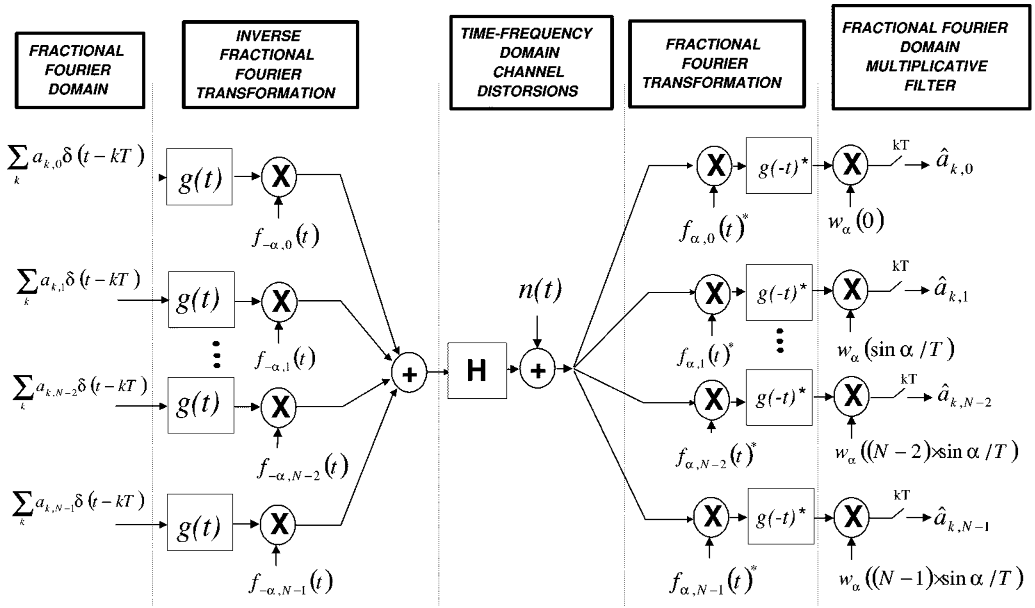

Fig. 4. Block diagram of an FRFT general multicarrier system. Observe that for $\alpha = \pi/2$ we obtain the traditional multicarrier system.

Notation: Observe that in the following derivations the notation $[\mathbf{v}]_k$ is used for the kth element of vector $\mathbf{v}$ and the notation $[\mathbf{M}]_{k,n}$ is used for the k, n element of the matrix $\mathbf{M}$.

Assume that $T = RT_s$ for R positive integer and write (1) as

$$s(t) = \sum_{k=-\infty}^{+\infty} \sum_{l=0}^{N-1} a_{k,l}\phi_l(t - kRT_s).$$

Sampling at $T_e \le T_s$, we obtain

$$s_k(n) = \sum_{m=-\infty}^{+\infty} \sum_{l=0}^{N-1} a_{m,l}\phi_l((nN - mR)T_s - kT_e)$$

with $s_k(n) = s((nR - k)T_e)$, $k = 0, 1, \ldots N - 1$, where R is such that $RT_e = NT_s$. The particular case $R = N$ and $T_e = T_s$, $\phi_l(t) = g(t)e^{2\pi j f_l t}$,

$$g(t) = \begin{cases} \dfrac{1}{\sqrt{NT_s}} & 0 \le t \le NT_s \\ 0 & \text{elsewhere} \end{cases}$$

and $f_l = l/NT_s$ collapses to the traditional FFT-based multicarrier system

$$\begin{aligned} s_k(n) &= \sum_{m=-\infty}^{+\infty} \sum_{l=0}^{N-1} a_{m,l}\, g((n-m)NT_s - kT_s) \\ &\quad \cdot \exp\left\{\frac{j2\pi lk}{N}\right\} \\ &= \frac{1}{\sqrt{NT_s}} \sum_{l=0}^{N-1} a_{n-1,l} \exp\left\{\frac{j2\pi lk}{N}\right\}. \end{aligned} \tag{18}$$

If we define $\mathbf{a}(n) = [a_{n-1,0}, a_{n-1,1}, \ldots, a_{n-1,N-1}]^T$ and $\mathbf{s}(n) = [s_0(n), s_1(n), \ldots, s_{N-1}(n)]^T$, (18) can be rewritten in vector form as

$$\mathbf{s}(n) = \mathbf{F}^{[-1]}\mathbf{a}(n) \tag{19}$$

where $\mathbf{F}^{[-1]}$ represents the (orthonormal) mapping in the traditional Fourier domain and we have assumed without loss of generality a unitary sample period T_s.

The instantaneous frequency of the nth waveform in (7) is $(-t\cot\alpha/2\pi) + (n/T)$. So it is evident that, since $|t| \le T/2$ and since $n \le N - 1$ (because we want to use N of these functions), the maximum frequency extent of the signal

$$s_k(t) = \sum_{n=0}^{N-1} a_{k,n} f_{-\alpha,n}(t) \tag{20}$$

is $(T\cot\alpha/4\pi) + ((N-1)/T)$ (if the symbols $a_{k,n}$ are assumed identically and independently distributed, i.i.d.). Limiting the value of α in the range $0.5\pi/2 \le |\alpha| \le 1.5\pi/2$, we also have that the maximum frequency for $s_k(t)$ is

$$\frac{T|\cot\alpha|}{4\pi} + \frac{N-1}{T} \le \frac{F_{\max}}{2} = \frac{N}{2T}$$

for any value of N, T. The relationship between T and N is enforced by the fact that the time–frequency product of $s_k(t)$ must be equal to N and that the sample period is chosen in such a way that $T = \sqrt{N}T_s$. If this is the case, $F_{\max} = (\sqrt{N}/T_s)$ and the signal set occupies an "equal" portion of time and frequency.

Now we try to obtain a formula for the discrete-time computation of (16). Since we need $\mathcal{F}_\alpha\{r_k(t)\}(u)$, we study the problem of obtaining $\mathcal{F}_\alpha\{s_k(t)\}(u)$ from the definition

$$\begin{aligned} \mathcal{F}_\alpha\{s_k(t)\}(u) = A_\alpha e^{j\pi\phi_\alpha u^2} \cdot \int_{-\infty}^{+\infty} & e^{-j2\pi\beta_\alpha tu} \\ & \cdot e^{j\pi\phi_\alpha t^2} s_k(t)\, dt \end{aligned} \tag{21}$$

for $\phi_\alpha = \cot\alpha$ and $\beta_\alpha = \csc\alpha$. The frequency "shearing" for the signal $s_k(t)$ when multiplied by the chirp $e^{j\pi\phi_\alpha t^2}$ for $0.5\pi/2 \le |\alpha| \le 1.5\pi/2$ results in a bandwidth which is at most

$2F_{\max}$ (contained in $[+F_{\max},\, -F_{\max}]$). So we can represent the signal $e^{j\pi\phi_\alpha t^2} s_k(t)$ using the Shannon interpolation formula

$$e^{j\pi\phi_\alpha t^2} s_k(t) = \sum_{n=-N}^{N} e^{j\pi\phi_\alpha (n/2F_{\max})^2} s_k\left(\frac{n}{2F_{\max}}\right) \cdot \operatorname{sinc}\left(2F_{\max}\left(t - \frac{n}{2F_{\max}}\right)\right) \tag{22}$$

where $N = (F_{\max}^2/T_s^2)$, and the summation is between N and $-N$ because $s_k(t)$ is bandlimited to $[+F_{\max}/2,\, -F_{\max}/2]$. Substitution of (22) into (21) and performing the integration[3] as in [11] gives

$$\mathcal{F}_\alpha\{s_k(t)\}(u) = \frac{A_\alpha}{2F_{\max}}\, e^{j\pi\phi_\alpha u^2} \cdot \sum_{n=-N}^{N} e^{-j2\pi\beta_\alpha (un)/(2F_{\max})} \cdot e^{j\pi\phi_\alpha (n/2F_{\max})^2} s_k\left(\frac{n}{2F_{\max}}\right)$$

which can be also discretized as

$$\begin{aligned}\mathcal{F}_\alpha\{s_k(t)\}\left(\frac{m}{2T}\right) &= \frac{A_\alpha}{2F_{\max}} \cdot \sum_{n=-N}^{N} \\ &\cdot \exp\left(j\pi\phi_\alpha\left(\frac{m}{2T}\right)^2 - j\beta_\alpha \frac{2\pi m}{2T}\frac{n}{2F_{\max}} + j\pi\phi_\alpha\left(\frac{n}{2F_{\max}}\right)^2\right) s_k\left(\frac{n}{2F_{\max}}\right).\end{aligned}$$

Using the relations among T, $F_{\max}$, and N with again a unitary sample period T_s, we obtain the operator for the fractional Fourier domain operator

$$[\mathbf{K}_\alpha]_{m,n} = K_\alpha(m-N-1,\, n-N-1) \qquad m,\, n = 1, 2, \ldots 2N+1$$

with

$$K_\alpha(m,n) = \frac{A_\alpha}{2\sqrt{N}} \exp\left\{j\pi\phi_\alpha \frac{m^2}{4N} - j2\pi\beta_\alpha \frac{mn}{4N} + j\pi\phi_\alpha \frac{n^2}{4N}\right\}.$$

A multicarrier system in the (discrete) fractional Fourier domain can be proposed that uses the operator

$$\mathbf{F}^{[a]} = \mathbf{D}\mathbf{K}_{a(\pi/2)}\mathbf{J} \tag{23}$$

at the receiver. The matrix $\mathbf{D}$ is a decimation by two matrices[4] and $\mathbf{J}$ is an interpolation by two matrices[5] [14]. At the transmitter, a natural choice is to replace $\mathbf{F}^{[-1]}$ with $\mathbf{F}^{[-a]}$ in (19). We then obtain the desired block operation

$$\mathbf{s}(n) = \mathbf{F}^{[-a]}\mathbf{a}(n).$$

Observe that this mapping is valid[6] for $0.5 \le |a| \le 1.5$. The operator is such that

$$\mathbf{F}^{[a]}\mathbf{F}^{[-a]} \simeq \mathbf{I}$$

and the Fourier domain which results in the traditional FFT-based scheme is obtained for $a = 1$.

Remark: The described computational procedure results in an approximately orthogonal digital transformation. Nevertheless, several other schemes recently proposed obtain exact orthogonality.

The operator $\mathbf{F}^{[a]}$ allows an algorithm implementation with complexity $\mathcal{O}[N \log N]$, which is a remarkable property [11]. This is easily seen rewriting the block operation performed by the operator $\mathbf{F}^{[-a]}$ as

$$\mathbf{s}(n) = \mathbf{D}\mathbf{K}_{-a(\pi/2)}\mathbf{J}\mathbf{a}(n) = \mathbf{D}\tilde{\mathbf{a}}_{-a}(n)$$

where $\tilde{\mathbf{a}}_{-a}(n)$ can be obtained as

$$\begin{aligned}[\tilde{\mathbf{a}}_{-a}(l)]_{m+N+1} &= \sum_{n=-N}^{N} K_{-\alpha}(m,n)[\mathbf{J}\mathbf{a}(l)]_{n+N+1} \\ &= \frac{A_{-\alpha}}{2\sqrt{N}} \cdot \sum_{n=-N}^{N} \exp\left\{j\pi\phi_{-\alpha}\frac{m^2}{4N} - j2\pi\beta_{-\alpha}\frac{mn}{4N} + j\pi\phi_{-\alpha}\frac{n^2}{4N}\right\} \cdot [\mathbf{J}\mathbf{a}(l)]_{n+N+1} \\ &= \frac{A_{-\alpha}}{2\sqrt{N}} \exp\left\{j\pi(\phi_{-\alpha} - \beta_{-\alpha})\frac{m^2}{4N}\right\} \sum_{n=-N}^{N} x_n y_{m-n}\end{aligned} \tag{24}$$

where

$$x_n = \exp\left\{j\pi(\phi_{-\alpha} - \beta_{-\alpha})\frac{n^2}{4N}\right\}[\mathbf{J}\mathbf{a}(l)]_{n+N+1}$$

and

$$y_m = \exp\left\{j2\pi\beta_{-\alpha}\frac{m^2}{4N}\right\}.$$

The expression (24) shows that the summation is a simple convolution of the elements of the vector $\mathbf{J}\mathbf{a}(l)$ premultiplied by a chirp sequence. The result of the convolution is again

[3]The interested reader is referred to [11] for further explanation on this derivation.

[4]A decimation by P matrix $\mathbf{D}$ acts on an NP-vector to give the decimated N-vector. $\mathbf{D}$ is obtained from the identity $NP \times NP$ matrix removing $P-1$ rows from consecutive P rows. A decimation by 2 ($P = 2$) matrix with $N = 3$ is

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

[5]An interpolation by P matrix acts on a N vector to give the upsampled and interpolated NP-vector. An interpolation by 2 ($P = 2$) matrix with $N = 3$ and h_n as the interpolation filter is

$$\mathbf{J} = \begin{bmatrix} h_0 & h_1 & 0 & 0 & 0 & 0 \\ 0 & h_0 & h_1 & 0 & 0 & 0 \\ 0 & 0 & h_0 & h_1 & 0 & 0 \\ 0 & 0 & 0 & h_0 & h_1 & 0 \\ 0 & 0 & 0 & 0 & h_0 & h_1 \\ 0 & 0 & 0 & 0 & 0 & h_0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

[6]Extensions of this interval can be easily determined as derived in [11].

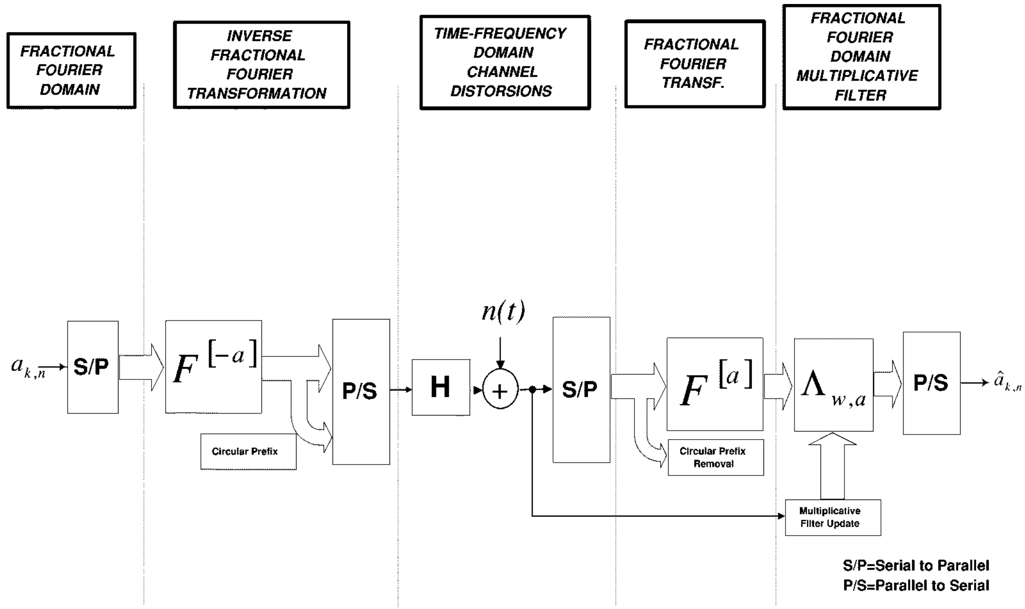

Fig. 5. Block diagram of the proposed FRFT-based multicarrier system. Observe that for $\alpha = \pi/2$ we obtain the traditional multicarrier system.

chirp-modulated. Evidently the majority of the computational effort is required by the convolution $\sum_{n=-N}^{N} x_n y_{m-n}$. A well-known efficient method to perform convolutions is the FFT which results in a complexity $\mathcal{O}[N \log N]$. Since the chirp modulations only contribute linearly to the computational load, the overall computational complexity is $\mathcal{O}[N \log N]$, approximately the same complexity required to implement the traditional N-carrier OFDM system based on the use of the FFT.

At the receiver, the signal block $\mathbf{r}(n)$, which is the sampled baseband equivalent of the channel-distorted signal $(\mathbf{H}s)(t) + n(t)$, is transformed back to the fractional Fourier domain and a multiplicative filter $\mathbf{w}_a$ is applied

$$\hat{\mathbf{s}}(n) = \mathbf{\Lambda}_{w,\,a}\mathbf{F}^{[a]}\mathbf{r}(n).$$

The matrix $\mathbf{\Lambda}_{w,\,a}$ is a diagonal $N \times N$ matrix whose diagonal elements consists of the elements of the complex weight vector $\mathbf{w}_a$. The solution that minimizes the mean squared error defined as

$$J = E\{||\mathbf{s}(n) - \hat{\mathbf{s}}(n)||^2\}$$

with $||\mathbf{x}(n)||^2 = \sum_i |[\mathbf{x}(n)]_i|^2$ obtained following the same methodology developed in [12] which gives

$$[\mathbf{w}_\alpha]_i = \frac{E\{[\mathbf{s}(n)]_i[\mathbf{r}^*(n)]_i\}}{E\{[\mathbf{s}(n)]_i,\,[\mathbf{r}^*(n)]_i\}}. \tag{25}$$

The block diagram of the discrete-time baseband FRFT-based system is shown in Fig. 5, where we also show the application of the circular prefix [17]. $\mathbf{F}^{[a]}$ (and $\mathbf{F}^{[-a]}$) is defined in (23). $\mathbf{\Lambda}_{w,\,a}$ is a diagonal matrix with complex diagonal elements. The traditional OFDM system is simply obtained by replacing the transformation $\mathbf{F}^{[-a]}$ with $\mathbf{F}^{[-1]}$ defined in (18) and (19) at the transmitter and $\mathbf{F}^{[a]}$ with $\mathbf{F}^{[-1]^H}$ at the receiver. The diagonal matrix $\mathbf{\Lambda}_{w,\,a}|_{a=1}$ then becomes a bank of single-tap frequency-domain equalizers.

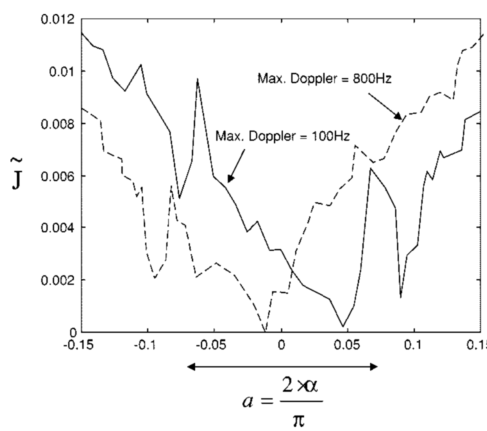

Fig. 6. The cost function $\tilde{J}$ versus $a = (2\alpha/\pi)$ shown in normalized form for two different channel environments.

V. Performance Assessment

In this section, we study the performance of the scheme in a wireless high mobility environment. The symbol period is 160 μs, the delay spread is up to 40 μs, modulation scheme is QPSK coherently detected, the DMT symbol duration (N) is 128, the carrier frequency is 5.2 GHz, the Doppler spread f_D is 200–1000 Hz, and the propagation environment is a nonline of sight (Rayleigh) RF link. We include 40 μs of guard time and assume a frame structure with 15 reference symbols dedicated to training of the multiplicative filter. The measure of performance is a raw (uncoded) bit-error rate (BER) averaged over 10 000 multicarrier blocks. The baseline performance is an OFDM system that employs the same parameters with traditional IFFT/FFT processing and the "ideal diagonalizing operator" which assumes perfect knowledge of the channel (both at the transmitter and at the receiver). In discrete time, a generic time-varying linear operator is represented by $\mathbf{H}$ and $\mathbf{H}$ is in fact a matrix with no "structure" (it becomes a very structured convolution matrix in the static case, easily diagonalized by the

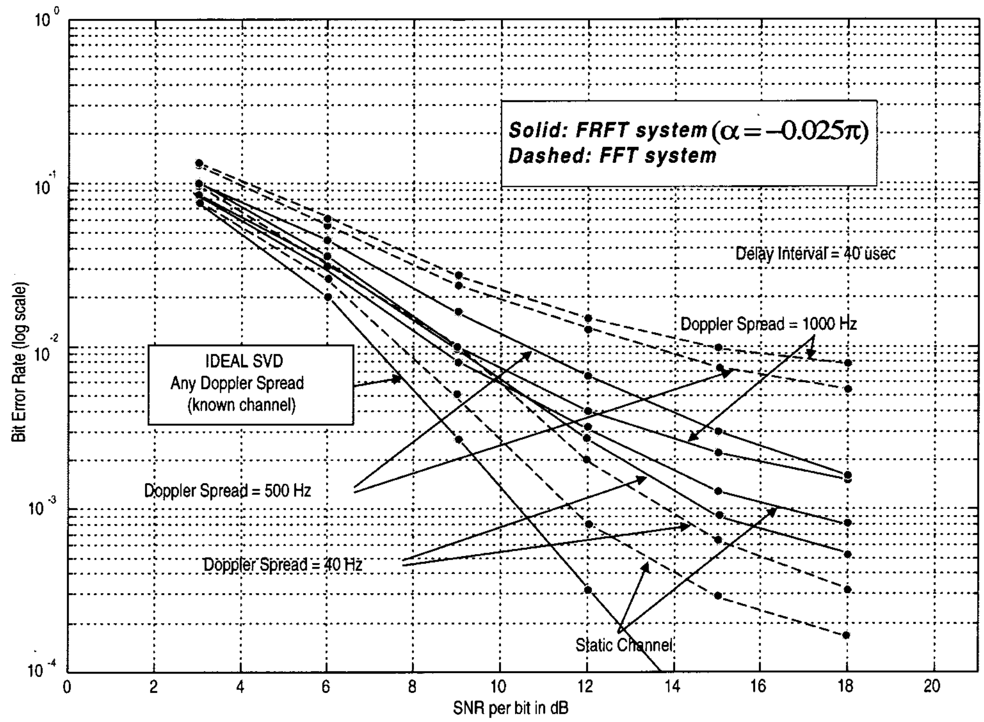

Fig. 7. BER performance of the fractional Fourier scheme as it compares to the classical traditional scheme based on IFFT/FFT processing at different Doppler spreads and to the ideal SVD-based system with a perfectly known channel.

FFT with cyclic prefix). Define the singular value decomposition of $\mathbf{H}$ as $\mathbf{H} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$, where $\mathbf{\Sigma}$ is diagonal. To diagonalize $\mathbf{H}$, one can use the right singular vectors $\mathbf{V}$ (instead of $\mathbf{F}^{[-a]}$) at the transmitter and the left singular vectors $\mathbf{U}^H$ (instead of $\mathbf{F}^{[a]}$) at the receiver (refer to Fig. 5). The optimum one-tap equalizers are given by the diagonal elements of $\mathbf{\Sigma}^{-1}$.

The channel model is based on the wide-sense stationary uncorrelated scattering (WSSUS) assumption [15]. The complex weights are generated as filtered Gaussian processes fully specified by the scattering function $S(t, \tau)$. In particular, each process has a frequency response equal to the square root of the Doppler power density spectrum.[7] The α angle (or equivalently the parameter a) for the fractional Fourier domain is selected to minimize the following cost function:

$$\begin{aligned}\tilde{J}(\alpha) = \sum_i & E\{[\mathbf{s}(n)]_i[\mathbf{s}^*(n)]_i\} \\ & - 2\mathrm{Re}\{[\mathbf{w}_\alpha]_i^* E\{[\mathbf{s}(n)]_i[\mathbf{r}^*(n)]_i\}\} \\ & + [\mathbf{w}_\alpha]_n[\mathbf{w}_\alpha^*]_n E\{[\mathbf{r}(n)]_i[\mathbf{r}^*(n)]_i\}. \qquad (26)\end{aligned}$$

Observe that the computational cost of computing $\tilde{J}(\alpha)$ is essentially equivalent to the cost associated with the calculation of $E\{[\mathbf{s}(n)]_i[\mathbf{r}^*(n)]_i\}$, $E\{[\mathbf{r}(n)]_i[\mathbf{r}^*(n)]_i\}$, and $E\{[\mathbf{s}(n)]_i[\mathbf{s}^*(n)]_i\}$ for each value of α [$\mathbf{s}(n)$ and $\mathbf{r}(n)$ evidently depend on α]. For a given Doppler spread, α can be optimized minimizing (26). Fig. 6 shows $\tilde{J}$ with respect to $a = (2\alpha/\pi)$, for a Rayleigh faded channel with $f_D = 100$ Hz and $f_D = 800$ Hz. There is an optimum value of α in each case. In the examples below, we optimize the value of the angle α for the fractional Fourier processing off-line.

Fig. 7 reports performance for $f_D = 0 - 1000$ Hz. Observe that as the rate of change in the channel decreases the performance improvement with respect to the FFT-based system also decreases. For a static channel, the classical scheme is more effective than the FRFT scheme. As we increase Doppler spread, the suboptimality of the FFT-based system is evidently outbalanced by the performance of the FRFT method.

Fig. 8 reports the performance for $f_D = 0 - 900$ Hz of the FRFT scheme, the FFT scheme, and the single-carrier QR-RLS DFE equalized scheme [16]. The SNR per bit is 10 dB and the delay spread is 20 μs. We observe an increased robustness with respect to the traditional FFT scheme. Improved performance for the FRFT scheme could be obtained optimizing α at each Doppler spread.

As already pointed out, we optimize off-line the α angle. However, an improvement can be obtained if we incorporate an online estimation procedure (not studied in this paper). We can envision a frame format organized in FRFT blocks, where in each frame a few blocks are dedicated to training of the multiplicative filter. Training the multiplicative filter requires the following:

- For each value of α quantized on a grid with $\mathcal{G}$ steps, transmission of the inverse FRFT (corresponding to each particular α) of $\mathcal{N}$ training signal blocks.
- Estimation of $E\{[\mathbf{s}(n)]_m, [\mathbf{r}(n)]_m^*\}$, $E\{[\mathbf{r}(n)]_m, [\mathbf{r}(n)]_m^*\}$ for each value of α in the grid.
- Compute (26) for each value of α in the grid and select α_{opt} as the value which minimizes $\tilde{J}$.
- Compute $\mathbf{w}_{\alpha_{\mathrm{opt}}}$ using (25).

[7]The Doppler spectrum is approximated by rational filtered processes. The filters are described by their 3-dB bandwidth which is called the normalized Doppler frequency. The additional assumption is that all channels and complex weights have the same Doppler spectrum.

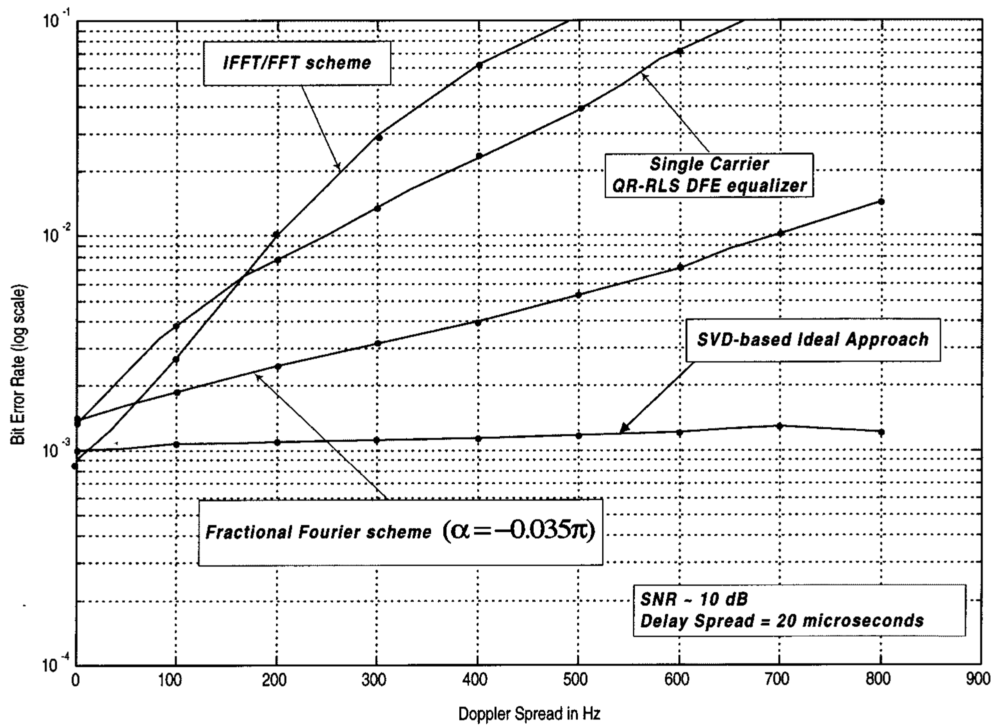

Fig. 8. BER performance versus Doppler spread for the fractional Fourier scheme as it compares to the classical traditional schemes based on IFFT/FFT processing, to the single carrier equalized scheme, and to the ideal SVD-based system with perfectly known channel.

VI. Conclusion

In this work, we have introduced the idea that frequency-varying basis functions are more suitable for multicarrier transmission over rapidly time-varying channels with respect to traditional frequency-invariant carriers, traditionally employed in OFDM systems. While the selection of a channel-dependent signal basis in the case of a static channel is well understood and corresponds to basis functions equal to the eigenmodes of the channel autocorrelation (well approximated by Fourier bases), the exact implementation of an eigendecomposition in the time-varying case is very difficult. In fact, the eigenmodes of a time-varying channel will be frequency-varying. The basic concept we have disclosed is based on a chirp-like harmonic decomposition for the RF propagation channel that matches the essential time-varying characteristics of the system. Using recently introduced schemes for fractional Fourier signal analysis, we have shown that, at no extra computational cost, it is possible to obtain a significant performance improvement in rapidly fading channels. The proposed methodology is an approximation to the optimum nonstationary approach which, in channels characterized by large Doppler spread, remarkably outperfoms the classical FFT-based scheme. Starting from a continuous time description, we have derived a discrete-time implementation amenable to digital processing and shown that the computational complexity for a block of size N is equivalent to the well-known $N \log N$ complexity of FFT algorithms.

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