Minimum Detectable Bias

Max Martone, MRT, Inc., 07/12/2007, www.martoneradiotech.com

When we attempt to isolate a fault in a certain measurement set we are trying simply to detect and identify a biased measurement in the set. How small of a bias we can reliably detect and identify is dictated by the expected noise on the measurements and by the geometry of the positioning problem.

The Minimum Detectable Bias is an important quantity.

Assume we have a set with n measurements made of a combination of:

- Time of Arrival: TOA,
- Time Difference of Arrival: TDOA,
- Angle of Arrival: AOA,

and that we are trying to resolve for k dimensions ($k = 2$ for ground applications or $k = 3$ for airborne cases).

$\mathbf{J}$ is the matrix that contains the linearized equations for the position problem

$$\mathbf{J}\delta\mathbf{x} = \delta\mathbf{f},$$

$\delta\mathbf{x}$ is the coordinate correction, and $\delta\mathbf{f}$ is the measurement correction.

Assume that the probability of having one bad measurement is α and that α is so small that the probability of having two bad measurements is negligible.

The statistical model is

$$\delta\mathbf{f} = \delta\mathbf{f}_0 + \mathbf{b} + \mathbf{e}, \tag{1}$$

where

- $\mathbf{e}$ is a multivariate Gaussian process with covariance $\mathbf{K}_e$, and mean $\mu_e = \mathbf{0}$,
- $\mathbf{b}$ is a deterministic, but unknown bias assumed to have the form $\mathbf{b} = \mathbf{f}_i \cdot b$ with $\mathbf{f}_i$ is a column vector made of all zeros and a 1 in the ith element,
- $\delta\mathbf{f}_0$ is the nominal noise-less measurement.

Define the two projection matrices

$$\mathbf{P} = \mathbf{J}\left[\mathbf{J}^T\mathbf{K}_e^{-1}\mathbf{J}\right]^{-1}\mathbf{J}^T\mathbf{K}_e^{-1},$$

and

$$\mathbf{Q} = \mathbf{I} - \mathbf{P}.$$

The matrix $\mathbf{Q}$ is sometimes called sensitivity matrix and it is extremely important because it relates the measurements to the fault (or residual) vector

$$\mathbf{r} = \mathbf{Q}\cdot\delta\mathbf{f}. \tag{2}$$

The classical statistic for fault detection is

$$D = \mathbf{r}^T\mathbf{r} \;\underset{H_1}{\overset{H_0}{\lessgtr}}\; T,$$

where T is a threshold, H_0 is the no-fault hypothesis and H_1 is the fault hypothesis.

This is a *power test*.

D has central chi-square ditribution under the null hypothesis (no fault) and non-central chi-square under the fault hypothesis (if measurements can be assumed Gaussian).

Minimum Detectable Bias

The bias that we can reliably detect depends on the noise characteristics on the measurements and on the geometry of the positioning problem. If we assume that $\mathbf{K}_e = \sigma_e^2\mathbf{I}$, the probability of not being able to detect a fault (in radar terminology the missed detection) is

$$P_e = \frac{1}{n}\sum_{i=1}^{n} p_1\left(T/\sigma_e^2|n-k, \frac{b^2}{\sigma_e^2}\mathbf{Q}_{i,i}\right) \tag{3}$$

where $p_1(\chi^2|m,\theta)$ is the non-central chi-square distribution with r degrees of freedom and θ is the non-centrality parameter.

The threshold T depends on the probability of false alarm (that is the probability of not detecting H_0) which we define P_0 by means of the following expression

$$P_0 = 1 - p_0(T/\sigma_e^2|n-k), \tag{4}$$

where $p_0(\chi^2|m)$ is the central chi-square distribution.

The well known expressions for $p_0(\chi^2|m)$ and $p_1(\chi^2|m,\theta)$ are

$$p_0(\chi^2|m) = \left[2^{-m/2}\cdot\Gamma\left(m/2\right)\right]^{-1}\int_0^{\chi^2} t^{r/2-1}e^{-t/2}dt,$$

and

$$p_1(\chi^2|m,\theta) = \sum_{i=0}^{\infty} e^{-\theta/2}\frac{(\theta/2)^i}{i!}p_0(\chi^2|r+2\cdot i).$$

So the procedure is

- assign a certain P_0,
- find the threshold T that satisfies (4) with P_0,
- assign a certain P_e,
- find the bias b that satisfies (3) with P_e.

The b that satisfies (3) is the Minimum Detectable Bias.